Topological semantics of provability logic

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Japaridze's polymodal logic GLP

modalities: $[0]$, $[1]$, $[2]$, ...
Fix an arithmetical theory T such as PA.

$$[n]\varphi \quad := \quad \text{“}\varphi \text{ is provable from } T \text{ and some true } \Pi^0_n\text{-sentences”}$$
$$\langle n\rangle\varphi \quad := \quad \neg[n]\neg\varphi = \text{“}\varphi \text{ is n-consistent”}$$

Axioms of GLP

Axioms:

1. $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$
2. $[n]([n]\varphi \to \varphi) \to [n]\varphi$ (Löb's axiom)
3. $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for $m < n$ (Σ^0_{m+1}-completeness)
4. $[m]\varphi \to [n]\varphi$, for $m < n$ (monotonicity)

Rules: modus ponens, $\vdash \varphi \Rightarrow \vdash [n]\varphi$.

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Arithmetical completeness theorem

Proved by Japaridze in 1986, improved by Ignatiev in 1993.

Theorem
$GLP \vdash \varphi \iff T \vdash \varphi^*$, for every evaluation $*$ of the variables of φ to arithmetical sentences.

Remark: Japaridze interpreted [1] as the closure of [0] under non-nested applications of the ω-rule.

Further developments

Ignatiev (1993): analysis of the closed fragment of *GLP*; Craig's interpolation property; fixed point property.

Boolos (1993): another interpretation of *GLP*, a textbook exposition.

Beklemishev (2000–2004): uses of *GLP* in proof theory.

- a consistency proof for Peano arithmetic PA (Gentzen's theorem);
- characterizing Π^0_n-theorems of PA;
- the Worm principle.

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The problem of models of *GLP*

Fact: There is no nontrivial Kripke *frame* in which all the axioms of *GLP* are true. Hence, *GLP* is not complete w.r.t. any class of standard Kripke frames.

Problem: find some manageable semantics for *GLP*.

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Problem: find some manageable semantics for *GLP*.

Set-theoretic interpretation

Let X be a nonempty set, $\mathcal{P}(X)$ the bool. algebra of subsets of X.

Consider any operators $\delta_n : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X); \delta_0, \delta_1, \dots)$.

We interpret $\langle n \rangle$ as δ_n, boolean operations in the standard way.

Question: Can $(\mathcal{P}(X); \delta_0, \delta_1, \dots)$ be a model of GLP and, if yes, when?

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Set-theoretic interpretation, contd.

Let $v : \mathrm{Var} \to \mathcal{P}(X)$ be an assignment of subsets of X to propositional variables; v is extended to arbitrary GLP-formulas:

- $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$;
- $v(\neg\varphi) = X \setminus v(\varphi)$;
- $v(\langle n \rangle \varphi) = \delta_n(v(\varphi))$.

Define: $X \vDash \varphi$ if $v(\varphi) = X$, for any v.

$Log(X) := \{\varphi : X \vDash \varphi\}$ (the logic of $(X; \delta_0, \delta_1, \dots)$).

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Derived set operators

Let X be a topological space, $A \subseteq X$.
Derived set $d(A)$ of A is the set of limit points of A:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \; y \in U_x \cap A.$$

Fact. If $(X, \delta) \models GL$ then X bears a unique scattered topology τ for which $\delta = d_\tau$, that is, $\delta : A \longmapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, A is τ-closed iff $\delta(A) \subseteq A$.
Equivalently, $c(A) = A \cup \delta(A)$ is the closure of A.

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Scattered spaces

Definition (Cantor): X is scattered if every nonempty $A \subseteq X$ has an isolated point.

Cantor–Bendixson sequence:

$$X_0 = X, \quad X_{\alpha+1} = d(X_\alpha), \quad X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha, \text{if } \lambda \text{ is limit.}$$

Notice that all X_α are closed and $X_0 \supset X_1 \supset X_2 \supset \dots$

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Examples

- Left topology $\tau_{\prec}$ on a strict partial ordering $(X, \prec)$.
  $A \subseteq X$ is open iff $\forall x, y\, (y \prec x \in A \Rightarrow y \in A)$.

  Fact: $(X, \prec)$ is well-founded iff $(X, \tau_{\prec})$ is scattered.

- Ordinal Ω with the usual order topology generated by intervals (α, β) such that $\alpha < \beta$, $\alpha, \beta \in \Omega \cup \{\pm\infty\}$.

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Löb's identity = scatteredness

Simmons 74, Esakia 81

Löb's identity: $\Diamond A = \Diamond(A \wedge \neg\Diamond A)$.

Topological reading: $d(A) = d(A \setminus d(A)) = d(\mathsf{iso}(A))$,
where $\mathsf{iso}(A) = A \setminus d(A)$ is the set of isolated points of A.

Fact: The following are equivalent:

- X is scattered;
- $d(A) = d(\mathsf{iso}(A))$ for any $A \subseteq X$;
- $(X, d) \models \mathsf{GL}$.

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Completeness theorems

Theorem (Esakia 81): There is a scattered X such that $Log(X, d) = \mathsf{GL}$. In fact, X is the left topology on a countable well-founded partial ordering.

Theorem (Abashidze/Blass 87/91): Consider $\Omega \geq \omega^{\omega}$ with the order topology. Then $Log(\Omega, d) = \mathsf{GL}$.

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Topological models for GLP

We consider poly-topological spaces $(X; \tau_0, \tau_1, \dots)$ where modality $\langle n \rangle$ corresponds to the derived set operator d_n w.r.t. τ_n.

Definition: X is a *GLP*-space if

- τ_0 is scattered;
- For each $A \subseteq X$, $d_n(A)$ is τ_{n+1}-open;
- $\tau_n \subseteq \tau_{n+1}$.

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Basic example: Esakia space

Consider a bitopological space $(\Omega; \tau_0, \tau_1)$, where

- Ω is an ordinal;
- τ_0 is the left topology on Ω;
- τ_1 is the interval topology on Ω.

Fact (Esakia): $(\Omega; \tau_0, \tau_1)$ is a model of GLP_2, but not an exact one: the linearity axiom holds for $\langle 0 \rangle$, that is,

$$[0](\varphi \to (\psi \vee \langle 0 \rangle \psi)) \vee [0](\psi \to (\varphi \vee \langle 0 \rangle \varphi)).$$

Derivative topology

Let (X, τ) be a scattered space and let τ^+ denote the topology generated (as a subbase) by τ and $\{d_\tau(A) : A \subseteq X\}$.

We call τ^+ derivative topology. τ^+ is the coarsest topology τ_1 such that $(X; \tau, \tau_1)$ is a GLP$_2$-space.

Thus, any (X, τ) generates a *GLP*-space $(X; \tau_0, \tau_1, \ldots)$ with $\tau_0 = \tau$ and $\tau_{n+1} = \tau_n^+$, for each n.

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Completeness for GLP_2

GLP_2 is complete w.r.t. GLP_2-spaces generated from the left topology on a well-founded partial ordering (with Guram Bezhanishvili and Thomas Icard).

Theorem: There is a countable GLP_2-space X such that $Log(X, d_0, d_1) = GLP_2$.

In fact, X has the form $(X; \tau_{\prec}, \tau_{\prec}^{+})$ where $(X, \prec)$ is a well-founded partial ordering.

Aside: This seems to be the first naturally occurring example of a logic that is topologically complete but not Kripke complete.

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Difficulties

Difficulties for three or more operators.

Fact. If (X, τ) is hausdorff and first-countable (i.e. if each point has a countable neighborhood base), then (X, τ^+) is discrete.

Proof: Each $a \in X$ is a unique limit of a countable sequence $A = \{a_n\}$. Hence, $\{a\} = d(A)$ is open.

Ordinal GLP-spaces

Let τ_0 be the left topology on an ordinal Ω. It generates a GLP-space $(\Omega; \tau_0, \tau_1, \ldots)$. What are these topologies?

Fact: τ_1 is the order topology on Ω.

What is τ_2?

Club filter topology

Def. Let α be a limit ordinal.

- $C \subseteq \alpha$ is a club in α if C is τ_1-closed and unbounded below α.
- The filter generated by clubs in α is called the club filter. It is improper iff α has countable cofinality.

Fact. τ_2 is the club filter topology:

- τ_2-isolated points are ordinals of countable cofinality;
- if $cf(\alpha) > \omega$ then clubs in α form a neighborhood base of α;
- the least non-isolated point is ω_1.

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Stationary sets

Def. $A \subseteq \alpha$ is stationary in α if A intersects every club in α.

We have: $d_2(A) = \{\alpha : \ cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary$\}$

Remark: Set theorists call d_2 Mahlo operation.
Ordinals in $d_2(Reg)$, where Reg is the class of regular cardinals, are called weakly Mahlo cardinals. Their existence implies consistency of ZFC.

Stationary reflection

Studied by: Solovay, Harrington, Jech, Shelah, Magidor, and many more.

Def. Ordinal κ is *reflecting* if whenever A is stationary in κ there is an $\alpha < \kappa$ such that $A \cap \alpha$ is stationary in α.

Def. Ordinal κ is *doubly reflecting* if whenever A, B are stationary in κ there is an $\alpha < \kappa$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in α.

Theorem. κ is a τ_3-limit point iff κ is doubly reflecting.

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Mahlo topology τ_3

Fact (characterizing τ_3):

- If κ is not doubly reflecting, then κ is τ_3-isolated;
- If κ is doubly reflecting, then the sets $d_2(A) \cap \kappa$, i.e.,

  $$\{\alpha < \kappa : cf(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \alpha\},$$

  where A is stationary in κ, form a base of τ_3-open punctured neighborhoods of κ.

Analogy: Stationary sets at doubly reflecting cardinals play the role of clubs at ordinals of uncountable cofinality.

Consistency strength

Fact.

- If κ is weakly compact then κ is doubly reflecting.
- (Magidor) If κ is doubly reflecting then κ is weakly compact in L.

Cor. Assertion "τ_3 is non-discrete" is equiconsistent with the existence of a weakly compact cardinal.

Cor. It is consistent with ZFC that τ_3 is discrete and hence that GLP_3 is incomplete w.r.t. any ordinal space.

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Summary

Let θ_n denote the first limit point of τ_n.

	name	θ_n	$d_n(A)$
τ_0	left	1	$\{\alpha : A \cap \alpha \neq \varnothing\}$
τ_1	order	ω	$\{\alpha \in Lim : A \cap \alpha$ is unbounded in $\alpha\}$
τ_2	club	ω_1	$\{\alpha : cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary in $\alpha\}$
τ_3	Mahlo	θ_3	... ...

θ_3 is the first doubly reflecting cardinal.

On the location of the least non-isolated point

Definition. Let θ_n denote the first non-isolated point of τ_n (in the space of all ordinals).

We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, $\theta_3 =?$

ZFC does not know much about the location of θ_3:

- θ_3 is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, θ_3 need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model where $\aleph_{\omega+1}$ is doubly reflecting (Magidor);
- If θ_3 is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).

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Completeness of GLP$_2$ for Ω

A. Blass (91): 1) If $V = L$ and $\Omega \geq \aleph_\omega$, then GL is complete w.r.t. (Ω, τ_2). (Hence, "GL is complete" is consistent with *ZFC*.)

2) On the other hand, if there is a weakly Mahlo cardinal, there is a model of *ZFC* in which GL is incomplete w.r.t. (Ω, τ_2) (for any Ω).

(This is based on a model of Harrington and Shelah in which $\aleph_2$ is reflecting for stationary sets of ordinals of countable cofinality.)

Theorem (B., 2009): If $V = L$ and $\Omega \geq \aleph_\omega$, then GLP_2 is complete w.r.t. $(\Omega; \tau_1, \tau_2)$.

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Further topologies: a conjecture (for set-theorists)

Theorem (B., Philipp Schlicht): If κ is Π^1_n-indescribable, then κ is non-isolated w.r.t. τ_{n+2}. Hence, if Π^1_n-indescribable cardinals below Ω exist for each n, then all topologies τ_n are non-discrete.

Conjecture: If $V = L$ and Π^1_n-indescribable cardinals below Ω exist for each n, then GLP is complete w.r.t. Ω.

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Conjecture: If $V = L$ and Π^1_n-indescribable cardinals below Ω exist for each n, then GLP is complete w.r.t. Ω.

Topological completeness

GLP is complete w.r.t. (countable, hausdorff) *GLP*-spaces.

Theorem (B., Gabelaia 10): There is a countable hausdorff *GLP*-space X such that $Log(X) = GLP$.

In fact, X is ε_0 equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$.

If *GLP* complete w.r.t. a *GLP*-space X, then all topologies of X have Cantor-Bendixson rank $\geq \varepsilon_0$.

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In fact, X is ε_0 equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$.

If GLP complete w.r.t. a GLP-space X, then all topologies of X have Cantor-Bendixson rank $\geq \varepsilon_0$.

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Conclusions

1. The notion of *GLP*-space seems to fit very naturally in the theory of scattered topological spaces.

2. Connections between provability logic and infinitary combinatorics (stationary reflection etc.) are fairly unexpected and would need further study.

3. From the point of view of applications to the study of modal logics such as *GLP*, the models obtained are still 'too big' and not very handy.

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Thank you!