A Justification Logic with Common Knowledge

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Explicit Paradigms in Logic and Computer Science
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Justification Logic (Artemov 95)

modal logic	**justification logic**
$\Box A$	$[t]A$
A is known	t is evidence for A
A is provable	t is a proof for A

S4 and LP

S4 **axioms & rules**

$\Box(A \to B) \to (\Box A \to \Box B)$
$\Box A \to A$
$\Box A \to \Box\Box A$

modus ponens

$$\frac{A}{\Box A}$$

LP **axioms & rules**

$[t](A \to B) \to ([s]A \to [t \cdot s]B)$
$[t]A \to A$
$[t]A \to [!t][t]A$
$[t]A \vee [s]A \to [t + s]A$

modus ponens

$$\frac{}{[c]A}$$

where c is a constant
and A an axiom

Common Knowledge

Common Knowledge of A

everybody knows A and
everybody knows that everybody knows A and
everybody knows that everybody knows that everybody knows A
and
...

Another perspective

Common knowledge of A is the greatest fixed point of

$$\lambda X.\text{everybody knows } A \text{ and everybody knows } X.$$

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The system H_R (Fagin, Halpern, Moses, Vardi 95)

- All propositional tautologies
- $\Box_i(A \to B) \to (\Box_i A \to \Box_i B)$ (K)
- $\Box_i A \to A$ (T)
- $\Box_i A \to \Box_i \Box_i A$ (4)
- $CA \to E(A \wedge CA)$ (Co-CI)

and rules

$$\frac{A \quad A \to B}{B}\ (\text{MP}) \qquad \frac{A}{\Box_i A}\ (\text{Nec}) \qquad \frac{B \to E(A \wedge B)}{B \to CA}\ (\text{I-R})$$

The system H_{Ax} (Meyer, van der Hoek 95)

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- $\Box_i A \to \Box_i \Box_i A$ (4)
- $CA \to E(A \wedge CA)$ (Co-Cl)
- $C(A \to EA) \to (EA \to CA)$ (I-Ax)

and rules

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$$\frac{A}{CA} \text{ (C-Nec)}$$

A Justification Logic for Common Knowledge: General Idea

- take a copy of a justification logic for each agent
- add evidence terms for mutual and common knowledge
- operation to tuple individual evidence
- operation to access tuple
- operation to generate "infinite list"
- operations to split list into head and tail

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$$\mathsf{C}(A \rightarrow \mathsf{E}A) \rightarrow (\mathsf{E}A \rightarrow \mathsf{C}A)$$

- operations to split list into head and tail

$$\mathsf{C}A \rightarrow \mathsf{E}A \ , \qquad \mathsf{C}A \rightarrow \mathsf{EC}A$$

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$$[t]_{\mathsf{C}}(A \rightarrow [r]_{\mathsf{E}} A) \rightarrow ([s]_{\mathsf{E}} A \rightarrow [\mathsf{ind}(t, s)]_{\mathsf{C}} A)$$

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$$[t]_{\mathsf{C}} A \rightarrow [\mathsf{head}(t)]_{\mathsf{E}} A \ , \qquad [t]_{\mathsf{C}} A \rightarrow [\mathsf{tail}(t)]_{\mathsf{E}} [t]_{\mathsf{C}} A$$

Language

Let $\circledast \in \{1, \ldots, h, \mathsf{E}, \mathsf{C}\}$

$\mathsf{Cons}_{\circledast}$ a countable set of proof constants for $\circledast$

$\mathsf{Var}_{\circledast}$ a countable set of proof variables for $\circledast$

Prop a countable set of propositional variables.

Operations on Terms

- application and sum (for agents and C)
- positive inspection (for agents)
- tupling and projection (relates individual and mutual knowledge)
- induction and co-closure (relates mutual and common knowledge)

Formulae

$A ::= P_j \mid \bot \mid A \to A \mid [t]_{\circledast} A$ where $t \in \mathsf{Tm}_{\circledast}$

$\mathrm{LP}_h^{\mathsf{C}}$: A Justification Logic with Common Knowledge

Axioms

1. all propositional tautologies
2. $[t]_*(A \to B) \to ([s]_* A \to [t \cdot s]_* B)$ (application)
3. $[t]_* A \vee [s]_* A \to [t + s]_* A$ (sum)
4. $[t]_i A \to A$ (reflexivity)
5. $[t]_i A \to [!t]_i\, [t]_i A$ (inspection)
6. $[t_1]_1 A \wedge \cdots \wedge [t_h]_h A \to [\langle t_1, \ldots, t_h \rangle]_{\mathsf{E}} A$ (tupling)
7. $[t]_{\mathsf{E}} A \to [\pi_i t]_i A$ (projection)
8. $[t]_{\mathsf{C}} A \to [\mathrm{head}(t)]_{\mathsf{E}} A, \quad [t]_{\mathsf{C}} A \to [\mathrm{tail}(t)]_{\mathsf{E}}\, [t]_{\mathsf{C}} A$ (co-closure)
9. $[t]_{\mathsf{C}}(A \to [r]_{\mathsf{E}} A) \to ([s]_{\mathsf{E}} A \to [\mathrm{ind}(t, s)]_{\mathsf{C}} A)$ (induction)

where $* \in \{1, \ldots, h, \mathsf{C}\}$.

Rules *modus ponens* and *axiom necessitation*

$$\frac{A \quad A \to B}{B}\ (\mathrm{MP})\ , \qquad \frac{}{[c]_{\mathsf{C}} D}\ (\mathrm{AN})\ ,$$

where $c \in \mathrm{Cons}_{\mathsf{C}}$ and D is an axiom from the list above.

Constructive Necessitation and Induction Rule

Theorem

If $\vdash A$, *then, for each* $\circledast$ *there exists a ground term* $t_{\circledast}$ *such that*

$$\vdash [t]_{\circledast} A \ .$$

Corollary

If $\vdash B \to [s]_{\mathsf{E}}(A \wedge B)$, *then there exists a ground term* t *such that*

$$\vdash B \to [t]_{\mathsf{C}} A \ .$$

Definable Operations

Lemma

The following operations can be defined

- $[t]_{\mathsf{E}}(A \to B) \to ([s]_{\mathsf{E}} \to [t \cdot_{\mathsf{E}} s]_{\mathsf{E}} B)$,
- $[t]_{\mathsf{E}} A \vee [s]_{\mathsf{E}} A \to [t +_{\mathsf{E}} s]_{\mathsf{E}} A$,
- $[t]_{\mathsf{C}} A \to [\downarrow_i]_i A$,
- $[t]_{\mathsf{C}} A \to [\uparrow_i]_i [t]_{\mathsf{C}} A$,
- $[t]_{\mathsf{C}} A \to [!_{\mathsf{C}} t]_{\mathsf{C}} [t]_{\mathsf{C}} A$,
- $[t]_{\mathsf{C}} A \to [\Leftarrow t]_{\mathsf{C}} [\mathsf{head}(t)]_{\mathsf{E}} A$.

Epistemic Models

An epistemic model is a quadruple $\mathcal{M} = (W, R, \mathcal{E}, \nu)$, where (W, R, ν) is a Kripke model, i.e.,

- W is a non-empty set called possible worlds.
- $R\colon \{1, \ldots, h\} \to \mathcal{P}(W \times W)$ assigns a transitive and reflexive accessibility relation to each agent.
- $\nu\colon \mathsf{Prop} \to \mathcal{P}(W)$ is a truth valuation.

and

- $\mathcal{E}\colon W \times \mathsf{Tm} \to \mathcal{P}\left(\mathsf{Fm}_{\mathsf{LP}_h^{\mathsf{C}}}\right)$ is an evidence function determining the formulae evidenced by a term at a world.

Furthermore

- $R_{\mathsf{E}} := \bigcup_{i=1}^{h} R_i$,
- $R_{\mathsf{C}} := \bigcup_{n=1}^{\infty} (R_{\mathsf{E}})^n$, i.e. R_{C} is the transitive closure of R_{E}.

Satisfaction Relation

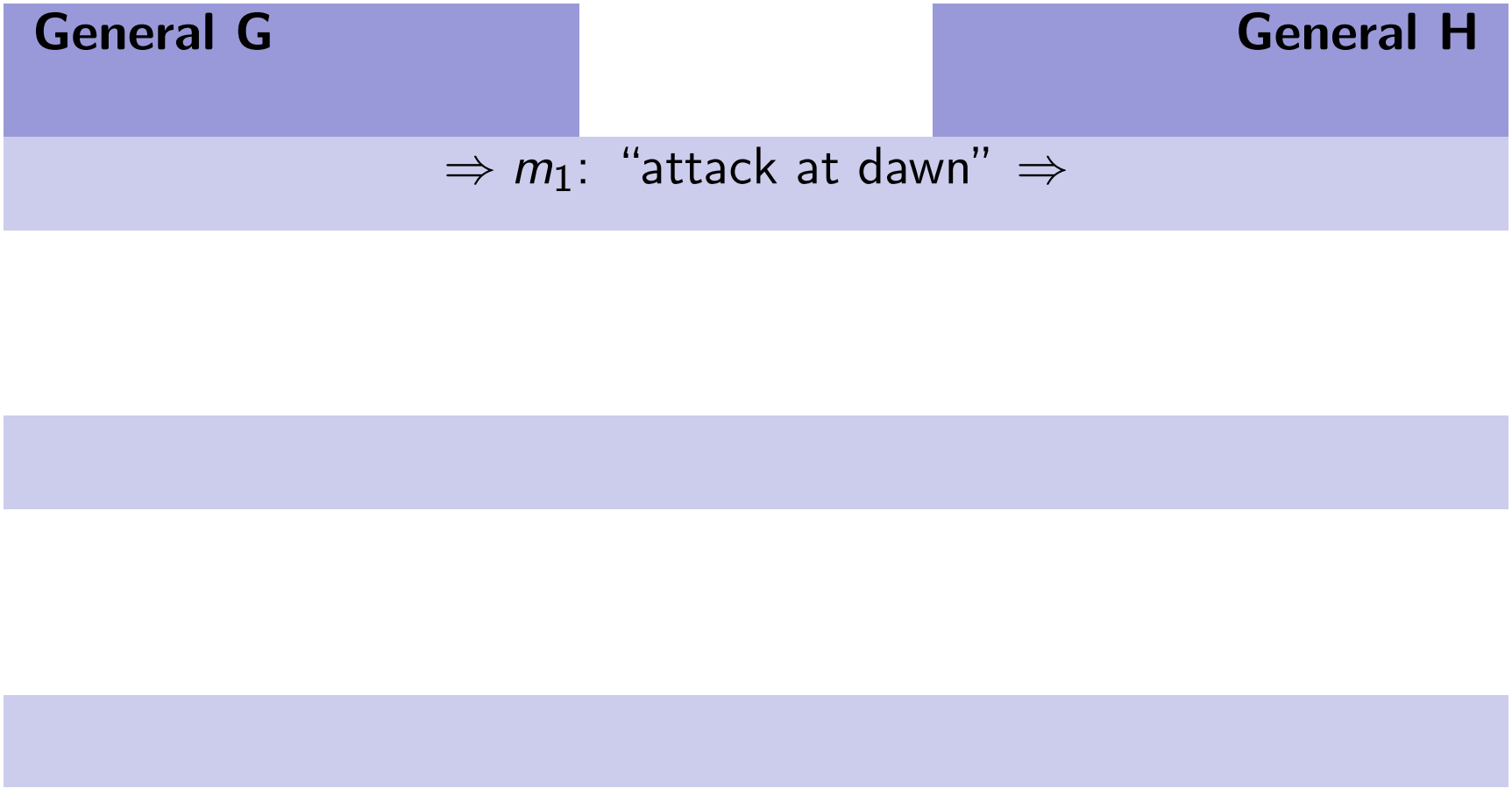
- $\mathcal{M}, w \Vdash P$ if and only if $w \in \nu(P)$;
- $\Vdash$ behaves classically with respect to propositional connectives;
- $\mathcal{M}, w \Vdash [t]_{\circledast} A$ if and only if
  1) $A \in \mathcal{E}_{\circledast}(w, t)$ and
  2) $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in R_{\circledast}$.

$[t]A$ if and only if

A is true in all possible worlds and t is admissible evidence for A.

Coordinated Attack

Coordinated Attack



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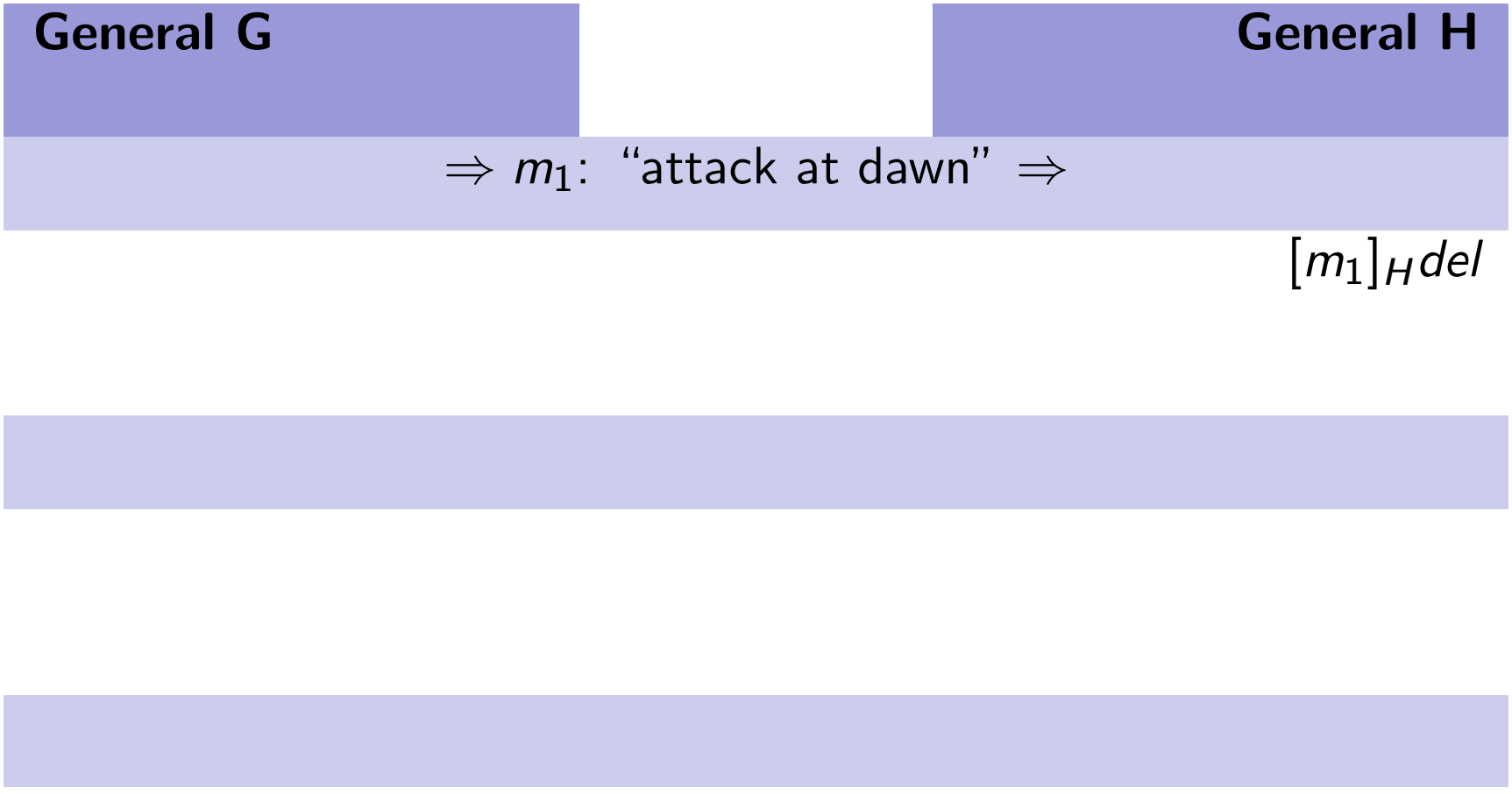

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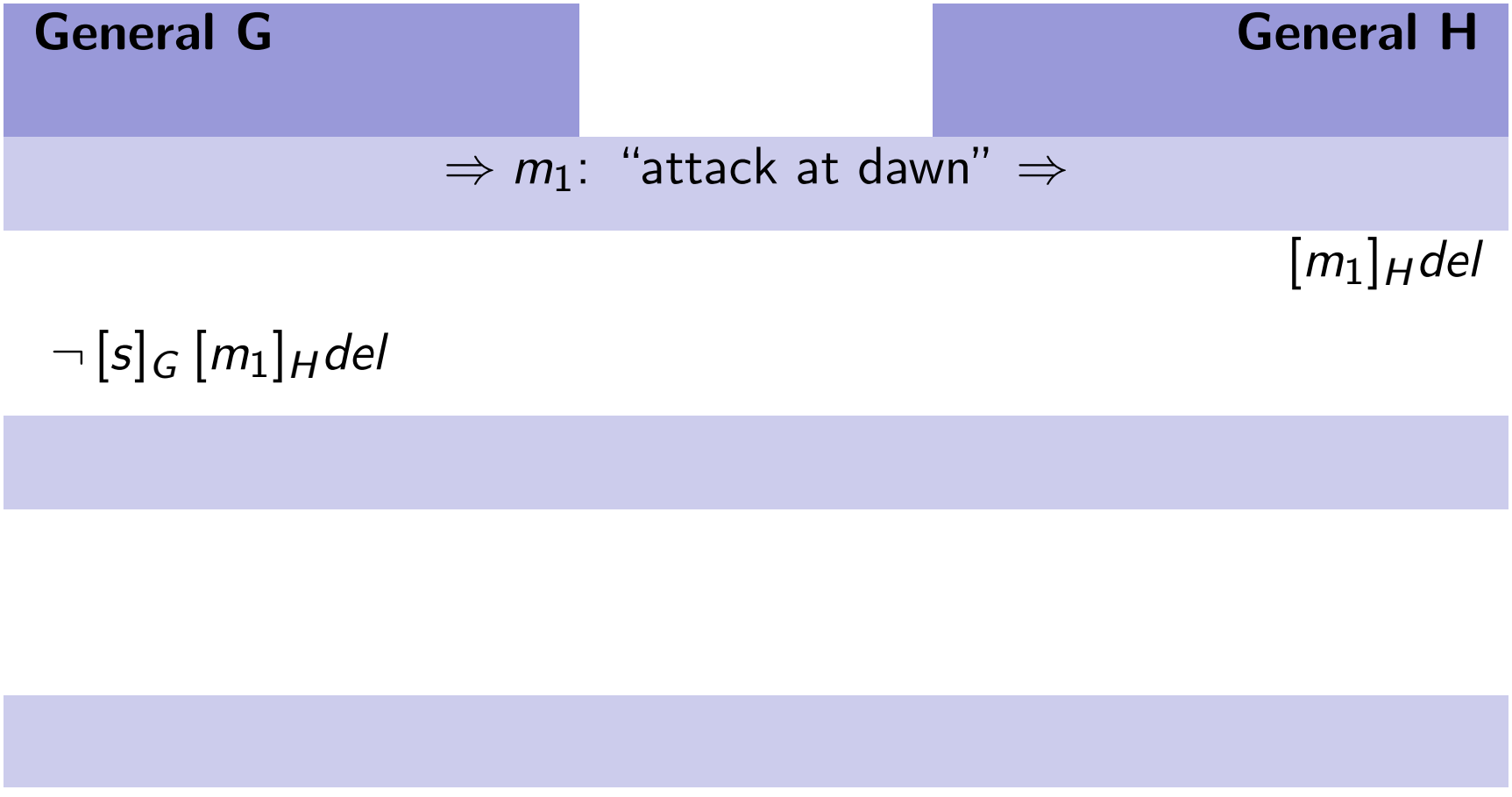

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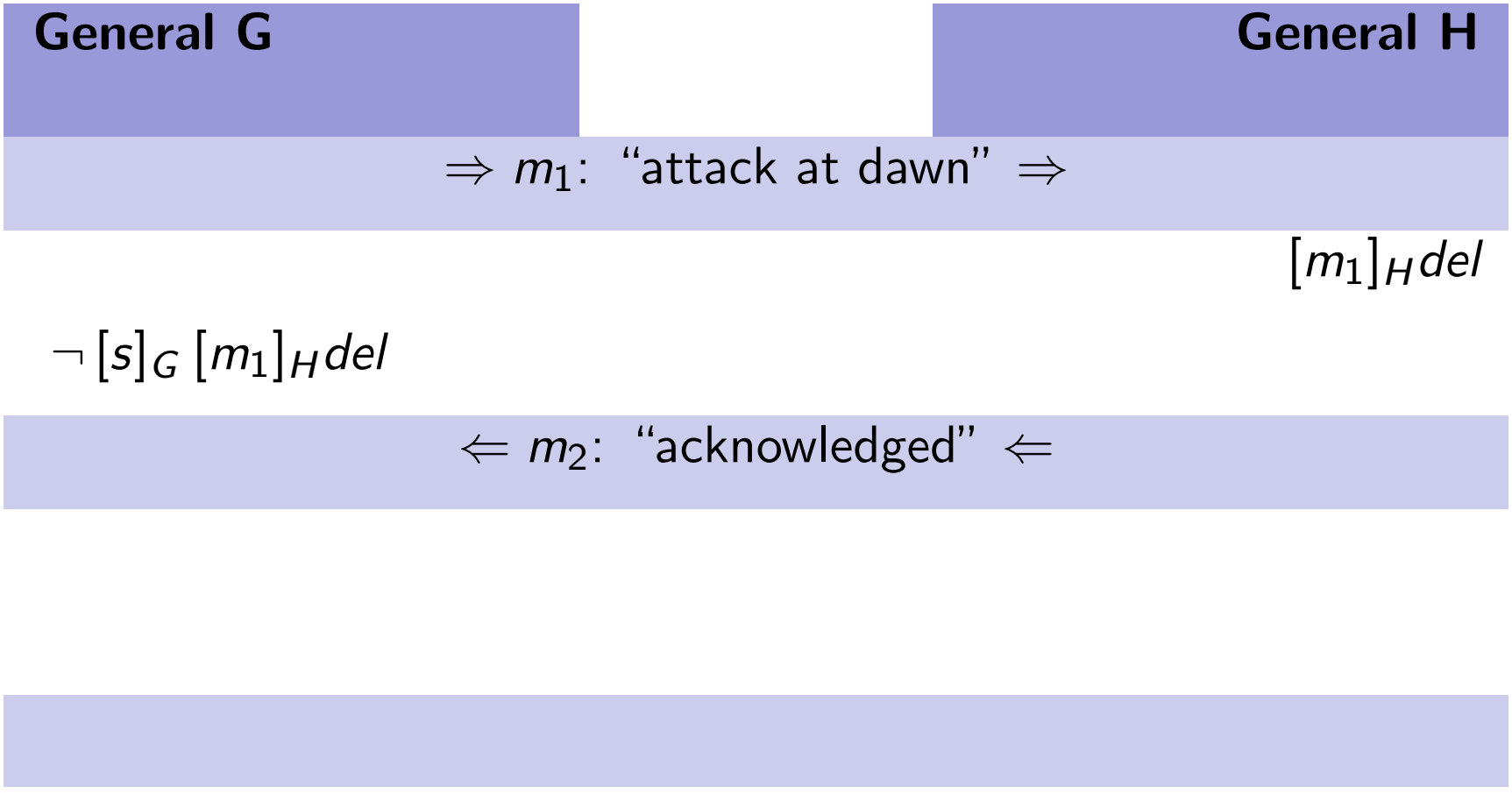

Coordinated Attack

General G | **General H**

$\Rightarrow m_1$: "attack at dawn" $\Rightarrow$

$[m_1]_H del$

$\neg [s]_G [m_1]_H del$

$\Leftarrow m_2$: "acknowledged" $\Leftarrow$

$[m_2]_G [m_1]_H del$

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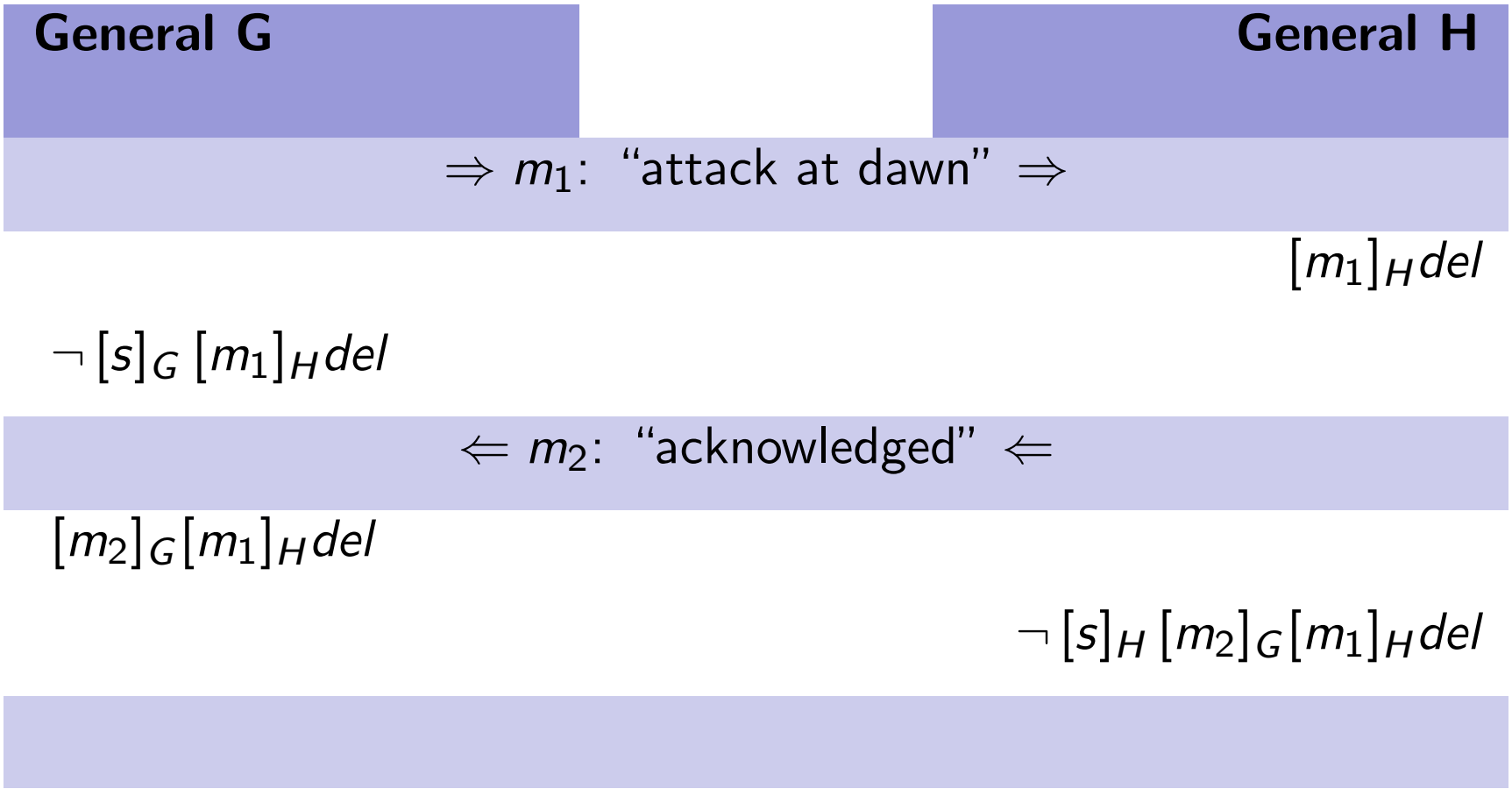

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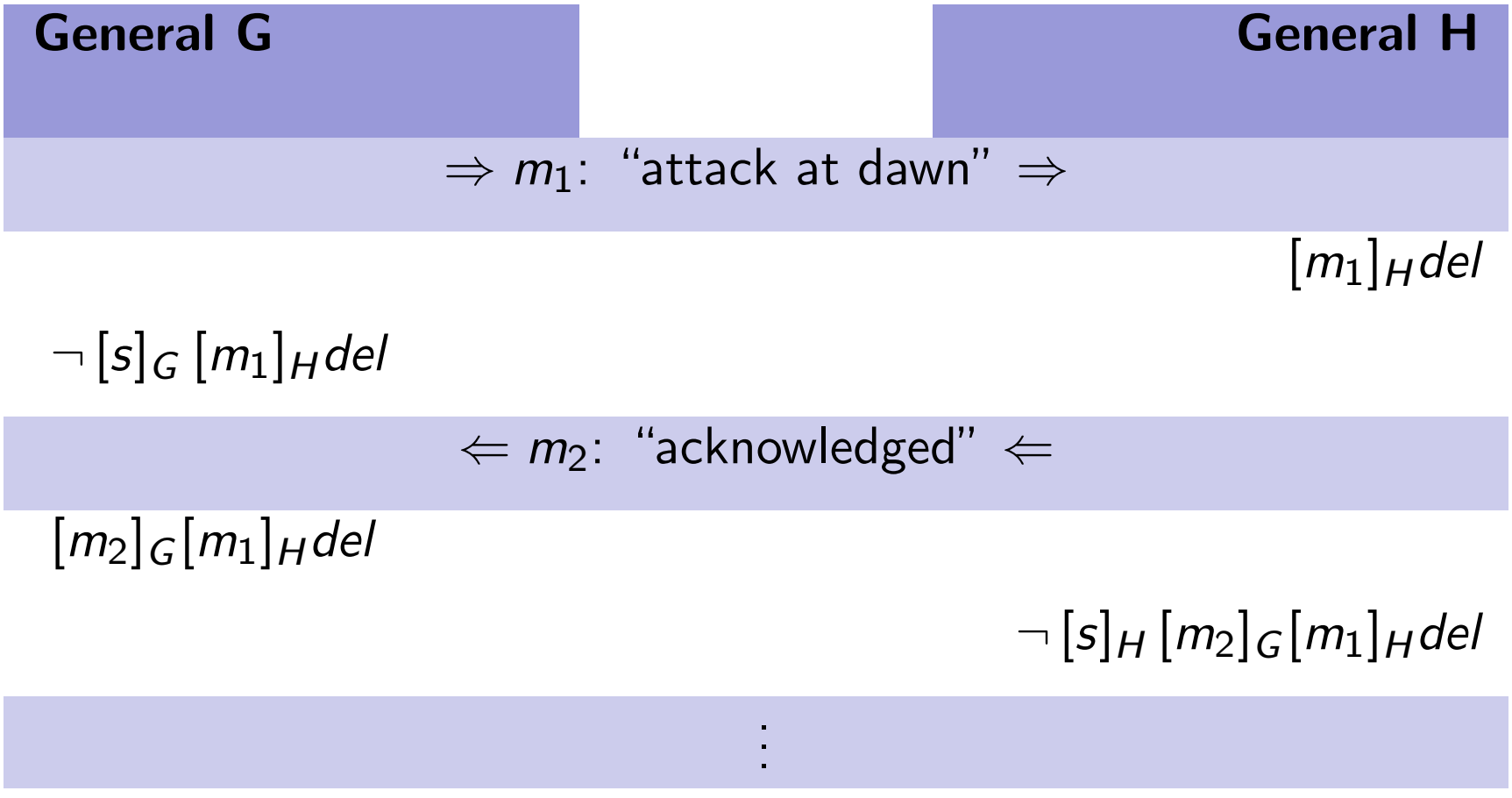

Soundness, Completeness and Decidability

Theorem

$\mathsf{LP}_h^{\mathsf{C}}$ *is sound and complete with respect to its class of epistemic models.*

Corollary

$\mathsf{LP}_h^{\mathsf{C}}$ *is complete with respect to the class of singleton models.*

Theorem

$\mathsf{LP}_h^{\mathsf{C}}$ *is decidable.*

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Conservativity

Theorem

$\mathsf{LP}_h^{\mathsf{C}}$ *is a conservative extensions of its corresponding multi-agent justification logic (without common knowledge).*

Corollary

$\mathsf{LP}_2^{\mathsf{C}}$ *is a conservative extension of Yavorskaya's minimal bimodal explicit evidence logic* LP_2.

Realization

Lemma
If $\mathsf{LP}_h^{\mathsf{C}} \vdash A$, *then* $\mathsf{S4} \vdash A^\circ$.

Question

What about the other direction?

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Fitting's Semantical Realization Technique

Observation

Fitting's technique relies only on a certain form of the canonical model, the Truth Lemma, and Constructive Necessitation.

Idea

If all possible realizations of a formula fail in a world of the canonical model, then also the forgetful projection of this formula must fail in the same world.

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Conclusions

- common knowledge is a natural addition to multi-agent justification logics
- alternative possibilities in analyzing epistemic scenarios
- major open problem: realization
- another interesting question: complexity
- long-term goal: dynamic epistemic logics
- more model theoretic tools
- justification logics without corresponding (normal) modal logics?