Operations and sets, constructively

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Bishop Constructive Mathematics

1967: Bishop's *Foundations of constructive analysis*

Two aspects of constructive mathematics Bishop style:

- it is fully compatibile with classical mathematics
- it is motivated by a computational attitude

Origins

1970's: Foundational systems for Bishop–style constructive mathematics

1. Intuitionistic set theory (Friedman '73, Myhill '73)
2. Explicit mathematics (Feferman '75)
3. Constructive type theory (Martin–Löf '75)
4. Constructive set theory (Myhill '75, Aczel '78)

CZF

Explicit mathematics and type theory are more faithful to Bishop's original motivation of making mathematics more computational

This is reflected by the explicit character of Feferman's theories and it is fully exploited in constructive type theory

Operational set theory wishes to combine some aspects of constructive set theory with some aspects of explicit mathematics

Constructive set theory

From a classical perspective we can see constructive set theory as obtained by a double restriction:

- Logic: Replacing classical with intuitionistic logic
- Further restraints to comply with a form of predicativity (usually termed generalised predicativity)

There is a fundamental difference with intuitionistic set theory which is fully impredicative (as it has full separation and powerset)

Constructive Zermelo Fraenkel set theory

CZF [Aczel78]	**ZF**
1. IFOLE	FOLE
2. Extensionality	Extensionality
3. Pair	Pair
4. Union	Union
5. Δ_0–separation	Separation
6. Fullness	Powerset
7. Strong collection	Replacement
8. Infinity	Infinity
9. Set induction	Foundation

IZF

Theorem [Aczel]: $\mathbf{CZF} + \mathbf{EM} = \mathbf{ZF}$

Constructive operational set theory

Let's look at the union axiom of **CZF**:

$$\forall a\, \exists x\, \forall y\, (y \in x \leftrightarrow \exists z \in a\, y \in z)$$

If we wish to implement **CZF** we might want to have an operation **un** which given the set a produces its union $\mathbf{un}\, a$

Can we have a constructive set theory where we have operations together with the usual sets?

Predecessors

- Intuitionistic set theory with rules: [Beeson88]
- Classical operational set theory: **OST** [Feferman06]
- Extensions of **OST**: [Jaeger07, Jaeger09, Jaeger09, JaegerZumbrunnen11]
- Constructive operational set theory: [CantiniCrosilla08, CantiniCrosilla10, Cantini11, CantiniCrosilla12]

Constructive Operational Set Theory

- Constructions as pairing, union, image, exponentiation, are perfectly good *operations* and we wish to represent them directly in our set theory
- We introduce *operations as rules* next to *functions as set–theoretic graphs*
- We have a notion of *application* for operations
- Operations are *non–extensional* while set–theoretic functions are *extensional*
- There is a limited form of *self–application*

The theory **ESTE**

Language: applicative extension, $\mathcal{L}^O$, of the usual first order language of Zermelo-Fraenkel set theory:

- $\in$, $=$, $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, $\forall$
- *App* (application)
- **K** and **S** (combinators)
- **el** (membership)
- **pair** , **un** , **im** , **sep** , **exp** (set operations)
- $\varnothing$, ω (set constants)

Application terms

We work within a definitional extension of $\mathcal{L}^O$ with application terms, defined as usual

(i) Each variable and constant is an application term

(ii) If t, s are application terms then ts is an application term

Abbreviations:

(i) $t \simeq x$ for $t = x$ when t is a variable or constant

(ii) $ts \simeq x$ for $\exists y \, \exists z \, (t \simeq y \, \wedge \, s \simeq z \, \wedge \, App(y, z, x))$

(iii) $t \downarrow$ for $\exists x \, (t \simeq x)$

(iv) $t \simeq s$ for $\forall x \, (t \simeq x \, \leftrightarrow \, s \simeq x)$

(v) $\varphi(t, \dots)$ for $\exists x \, (t \simeq x \, \wedge \, \varphi(x, \dots))$

(vi) $t_1 t_2 \dots t_n$ for $(\dots(t_1 t_2) \dots) t_n$

Conventions

A formula of $\mathcal{L}^O$ is Δ_0, iff
(a) all quantifiers occurring in it, if any, are bounded
(b) it does not contain *App*

Truth values: let $\bot := \varnothing$ and $\top = \{\varnothing\}$

The **class** of truth values: $\Omega := \mathcal{P}\top = \mathcal{P}\{\varnothing\}$

Further conventions

$f, g, \ldots$ for operations; $F, G, \ldots$ for set–theoretic functions

For a and b sets or classes, write

- $f : a \to b$ for $\forall x \in a\,(fx \in b)$
- $f : \mathbf{V} \to b$ for $\forall x\,(fx \in b)$, where $\mathbf{V} := \{x : x\downarrow\}$
- $f : a^2 \to b$ for $\forall x \in a\,\forall y \in a\,(fxy \in b)$
- $f : \mathbf{V}^2 \to b$ for $\forall x\,\forall y\,(fxy \in b)$ etc.

The theory **ESTE**

- Axioms and rules of first order intuitionistic logic with equality

Extensionality

- $\forall x\,(x \in a \leftrightarrow x \in b) \rightarrow a = b$

General applicative axioms

- $App(x, y, z) \wedge App(x, y, w) \rightarrow z = w$
- $\mathbf{K}xy = x \wedge \mathbf{S}xy{\downarrow} \wedge \mathbf{S}xyz \simeq xz(yz)$

Membership operation

- $\mathbf{el} : \mathbf{V}^2 \to \Omega$ and $\mathbf{el}\, xy \simeq \top \leftrightarrow x \in y$

Set constructors and Infinity

- $\forall x\, (x \notin \varnothing)$
- $\mathbf{pair}\, ab \downarrow \wedge \forall z\, (z \in \mathbf{pair}\, ab \leftrightarrow z = a \vee z = b)$
- $\mathbf{un}\, a \downarrow \wedge \forall z\, (z \in \mathbf{un}\, a \leftrightarrow \exists y \in a(z \in y))$
- $(f : a \to \Omega) \to \mathbf{sep}\, fa \downarrow \wedge \forall x\, (x \in \mathbf{sep}\, fa \leftrightarrow x \in a \wedge fx \simeq \top)$
- $(f : a \to V) \to (\mathbf{im}\, fa \downarrow) \wedge \forall x\, (x \in \mathbf{im}\, fa \leftrightarrow \exists y \in a(x \simeq fy))$
- $\mathbf{exp}\, ab \downarrow \wedge \forall x (x \in \mathbf{exp}\, ab \leftrightarrow (Fun(x) \wedge Dom(x) = a \wedge Ran(x) \subseteq b))$
- $Ind(\omega) \wedge \forall z\, (Ind(z) \to \omega \subseteq z)$

(i) For each term t, there exists a term $\lambda x.t$ with free variables those of t other than x and such that

$$\lambda x.t \downarrow \wedge (\lambda x.t)y \simeq t[x := y].$$

(ii) (Second recursion theorem) There exists a term **rec** with

$$\mathbf{rec}f \downarrow \wedge (\mathbf{rec}f = e \rightarrow ex \simeq fex).$$

Extensionality

Extensionality for sets:

$$\forall x\,(x \in a \leftrightarrow x \in b) \rightarrow a = b$$

Extensionality for operations:

$$\forall x\,(fx \simeq gx) \rightarrow f = g$$

Question: can operations be extensional?

Key results: I

- Operations are **non–extensional**: $\neg[\forall x\,(fx \simeq gx) \to f = g]$
- Application is **partial**: $\neg\forall x\,\forall y\,\exists z\,App(x, y, z)$
- **Bounded separation** has to be restricted to formulas **not containing App**
- The axiom of **choice** is problematic both for **set–theoretic functions** and for **operations**

Key results II: Proof–theoretic strength

ESTE has the same proof theoretic strength as **PA**

- *Lower bound*
  - **HA** is interpretable in **ESTE**
- *Upper bound*
  - We introduce an auxiliary theory **ECST*** and show that **ESTE** reduces to **ECST*** and the latter reduces to **PA**

ECSTS

ECST* is an extension of Aczel and Rathjen **ECST** by adding the exponentiation axiom

ECST is the subtheory of **CZF** with: extensionality, pair, union, Δ_0–separation, replacement, strong infinity

Note: no $\in$–induction is allowed
Rathjen: **ECST** is very weak: no number–theoretic sum

Upper bound

- Reduce **ESTE** to **ECST***: partial cut elimination and asymmetric interpretation
  - Sequent–style formulation of **ESTE** with active formulas positive in *App*
    A *partial cut elimination* theorem holds
  - *Asymmetric interpretation* of **ESTE** into **ECST***
    Idea: replace *App* by its finite stages App_n

Upper bound

- Reduce **ECST*** to **PA**: we introduce a classical theory of truth, $\mathbf{T_c}$, of the same strength as **PA** [Cantini96]
  - Translate **ECST*** in $\mathbf{T_c}$ by a realisability interpretation which recalls Aczel's interpretation of **CZF** in Constructive type theory
  - Here we need a separate rule for introducing the natural numbers (Rathjen's trick)

The picture

$$\mathbf{HA} \hookrightarrow \mathbf{ESTE} \hookrightarrow \mathbf{ECST}^* \hookrightarrow \mathbf{T_c} \hookrightarrow \mathbf{PA}$$

Friedman's **B**

Friedman's **B** [Friedman77]: set–theoretic foundation for constructive mathematics conservative over HA

Proposition: **B** is interpretable in **ESTE** + bounded dependent choice

Extensions of **ESTE**

- Andrea Cantini [Cantini11] has added a description operator to **ESTE** (conservative), and introduced impredicative extensions of **ESTE** with unbounded quantifiers and a fixed point operator

Transitive Closure

- TC: We add an operation τ that applied to a set a produces its transitive closure, τa

Transitive Closure

The theory $\mathbf{ESTE}_\tau$ is obtained from **ESTE** by adding a new constant τ to the language together with the axiom TC:

$$(\tau a{\downarrow} \wedge \mathit{Trans}(\tau a) \wedge a \subseteq \tau a \wedge (\forall c)(\mathit{Trans}(c) \wedge a \subseteq c \rightarrow \tau a \subseteq c))$$

where $\mathit{Trans}(z) := (\forall x)(\forall y)(x \in y \wedge y \in z \rightarrow x \in z)$

Key results

Theorem

ESTE$_\tau$ *is conservative over* **ESTE**

Idea of the proof: we make essential use of a separation between sets and natural numbers which is given in our model of the set–theoretic universe

By using $\mathbf{T_c}$'s axiom **GID** (Generalised Inductive Definitions) we can prove a useful induction principle which holds in the model, and, crucially, is acquired at no cost from a proof–theoretic perspective

We use the fixed point theorem of $\mathbf{T_c}$ and definition by cases on N to model the operator τ

$$\tau_{\mathbf{T_c}} a = \begin{cases} a, & \text{provided } a \text{ is a natural number;} \\ a \dot{\cup} \dot{\bigcup} \sup(\bar{a}, \lambda y.\tau_{\mathbf{T_c}}(\tilde{a} y)), & \text{provided } a \text{ is a set} \end{cases}$$

and use the induction principle to show that the model behaves as desired

Thank you!

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