The class NP

Isabel Oitavem

CMAF-UL and FCT-UNL

Recursion-theoretic approach

Theorem

$FPtime \simeq [\,\mathcal{B}\,;\ SC\,,\ SR_{\mathbb{W}}\,]$ *(Bellantoni-Cook 1992)*

$FPspace \simeq [\,\mathcal{B}\,;\ SC\,,\ TR_{\mathbb{W}}\,]$ *(O 2008)*

P $\subseteq$ NP $\subseteq$ Pspace

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P $\subseteq$ NP $\subseteq$ Pspace

FPtime $\subseteq$ $\cdots$ $\subseteq$ FPspace

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$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{Pspace}$

$\mathrm{FPtime} \subseteq \mathrm{FPtime} \cup \mathrm{NP} \subseteq \mathrm{FPspace}$

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$$\underbrace{\underbrace{\mathrm{FPtime}}_{\text{determ.}}} \subseteq \underbrace{\underbrace{\mathrm{FPtime} \cup \mathrm{NP}}_{\text{non-determ.}}} \subseteq \underbrace{\underbrace{\mathrm{FPspace}}_{\text{alternating}}}$$

Recursion-theoretic approach

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$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{Pspace}$

$\underbrace{\underbrace{\mathrm{FPtime}}_{\text{determ.}}}_{\text{recursion}} \subseteq \underbrace{\underbrace{\mathrm{FPtime} \cup \mathrm{NP}}_{\text{non-determ.}}}_{?} \subseteq \underbrace{\underbrace{\mathrm{FPspace}}_{\text{alternating}}}_{\text{tree-recursion}}$

Recursion-theoretic approach

$\mathrm{FPtime} \simeq [\, \mathcal{B}\, ;\, \mathrm{SC}\, ,\, \mathrm{SR}_{\mathbb{W}}\,]$ (Bellantoni-Cook 1992)

$\mathrm{FPspace} \simeq [\, \mathcal{B}\, ;\, \mathrm{SC}\, ,\, \mathrm{TR}_{\mathbb{W}}\,]$ (O 2008)

- $\mathbf{f} = \mathrm{SC}(\mathbf{g}, \bar{\mathbf{r}}, \bar{\mathbf{s}})$
  $f(\bar{x}; \bar{y}) = g(\bar{r}(\bar{x};\,); \bar{s}(\bar{x}; \bar{y}))$
- $\mathbf{f} = \mathrm{SR}_{\mathbb{W}}(\mathbf{g}, \mathbf{h})$
  $f(\epsilon, \bar{x}; \bar{y}) = g(\bar{x}; \bar{y})$
  $f(z0, \bar{x}; \bar{y}) = h(z0, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))$
  $f(z1, \bar{x}; \bar{y}) = h(z1, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))$
- $\mathbf{f} = \mathrm{TR}_{\mathbb{W}}(\mathbf{g}, \mathbf{h})$
  $f(p, \epsilon, \bar{x}; \bar{y}) = g(p, \bar{x}; \bar{y})$
  $f(p, z0, \bar{x}; \bar{y}) = h(p, z0, \bar{x}; \bar{y}, f(p0, z, \bar{x}; \bar{y}), f(p1, z, \bar{x}; \bar{y}))$
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Recursion-theoretic approach

FPtime $\simeq [\,\mathcal{B}\,;\ \mathrm{SC}\,,\ \mathrm{SR}_{\mathbb{W}}\,]$ (Bellantoni-Cook 1992)

NP $\simeq \cdots$

FPspace $\simeq [\,\mathcal{B}\,;\ \mathrm{SC}\,,\ \mathrm{TR}_{\mathbb{W}}\,]$ (O 2008)

- $\mathbf{f} = \mathrm{SC}(\mathbf{g}, \bar{\mathbf{r}}, \bar{\mathbf{s}})$
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Recursion-theoretic approach

$\text{FPtime} \simeq [\,\mathcal{B}\,;\ \text{SC}\,,\ \text{SR}_{\mathbb{W}}\,] = \text{ST}_0$ (Bellantoni-Cook 1992)

$\text{NP} \simeq [\,\text{ST}_0\,;\ \text{SC}_0\,,\ \cdots\,]$

$\text{FPspace} \simeq [\,\mathcal{B}\,;\ \text{SC}\,,\ \text{TR}_{\mathbb{W}}\,]$ (O 2008)

- $\mathbf{f} = \text{SC}_\mathbf{0}(\mathbf{g}, \bar{\mathbf{r}}, \bar{\mathbf{s}})$
  $f(\bar{x};\bar{y}) = g(\bar{r}(\bar{x};\,);\bar{s}(\bar{x};\bar{y})), \quad \bar{r},\bar{s} \in \text{ST}_0$
- $\mathbf{f} = \text{SR}_{\mathbb{W}}(\mathbf{g}, \mathbf{h})$
  $f(\epsilon,\bar{x};\bar{y}) = g(\bar{x};\bar{y})$
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$\text{FPtime} \simeq [\,\mathcal{B}\,;\ \text{SC}\,,\ \text{SR}_{\mathbb{W}}\,] \;=\; \text{ST}_0$ (Bellantoni-Cook 1992)
$\text{NP} \simeq [\,\text{ST}_0\,;\ \text{SC}_0\,,\ \cdots\,]$
$\text{FPspace} \simeq [\,\mathcal{B}\,;\ \text{SC}\,,\ \text{TR}_{\mathbb{W}}\,]$ (O 2008)

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- $\mathbf{f} = \text{TR}_{\mathbb{W}}(\mathbf{g}, \vee)$
  $f(p, \epsilon, \bar{x};\bar{y}) = g(p, \bar{x};\bar{y})$
  $f(p, z0, \bar{x};\bar{y}) = \vee(\quad ;\ f(p0, z, \bar{x};\bar{y}), f(p1, z, \bar{x};\bar{y}))$
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Recursion-theoretic approach

$\text{FPtime} \simeq [\,\mathcal{B}\,;\,\text{SC}\,,\,\text{SR}_{\mathbb{W}}\,] \;=\; \text{ST}_0$ (Bellantoni-Cook 1992)

$\text{NP} \simeq [\,\text{ST}_0\,;\,\text{SC}_0\,,\,\vee\text{-TR}^{L}_{\mathbb{W}}\,]$

- $\mathbf{f} = \text{SC}_{\mathbf{0}}(\mathbf{g}, \bar{\mathbf{r}}, \bar{\mathbf{s}})$
  $f(\bar{x};\bar{y}) = g(\bar{r}(\bar{x};\,);\bar{s}(\bar{x};\bar{y})), \quad \bar{r}, \bar{s} \in \text{ST}_0$

- $\mathbf{f} = \text{TR}_{\mathbb{W}}(\mathbf{g}, \vee) = \vee\text{-TR}^{\mathbf{L}}_{\mathbb{W}}(\mathbf{g})$
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- $\mathbf{f} = \text{SC}_{\mathbf{0}}(\mathbf{g}, \bar{\mathbf{r}}, \bar{\mathbf{s}})$
  $f(\bar{x}; \bar{y}) = g(\bar{r}(\bar{x};\,); \bar{s}(\bar{x}; \bar{y})), \quad \bar{r}, \bar{s} \in \text{ST}_0$

- $\mathbf{f} = \text{TR}_{\mathbb{W}}(\mathbf{g}, \vee) = \vee\text{-TR}^{\mathbf{L}}_{\mathbb{W}}(\mathbf{g})$
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$\quad\simeq [\,\text{ST}_0\,;\ \text{SC}_0\,,\ \vee\text{-TR}^{R}_{\mathbb{W}}\,]$

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  $f(\epsilon, \bar{x}; \bar{y}, p) = g(\bar{x}; \bar{y}, p)$

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Recursion-theoretic approach

Theorem

$NP \simeq [\, ST_0\, ;\, SC_0\, ,\, \vee\text{-}TR^{L}_{\mathbb{W}}\,]$

$\simeq [\, ST_0\, ;\, SC_0\, ,\, \vee\text{-}TR^{R}_{\mathbb{W}}\,]$

Lemma

For all $f \in [\, ST_0\, ;\, SC_0\, ,\, \vee\text{-}TR^{L}_{\mathbb{W}}\,]$ *there exists* $F \in [\, ST_0\, ;\, SC_0\, ,\, \vee\text{-}TR^{R}_{\mathbb{W}}\,]$ *such that*

$$\forall \bar{x} \forall \bar{y} \quad f(\bar{x}; \bar{y}) = F(\bar{x}, \bar{y};\,).$$