The Existence Property and Ordinal Analysis

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Formalization of intuitionistic logic

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Heyting 1930: intuitionistic predicate logic and arithmetic

Negative translation: **Kolmogorov** 1925, **Gentzen** and **Gödel** 1933.

Kleene's 1945 realizability for **HA**

a realizer of	**has the form**
A atomic	any e providing A is true.
$A \wedge B$	(a, b), where a is a realizer of A and b is a realizer of B.
$A \vee B$	$(0, a)$, where a is a realizer of A, or $(1, b)$, where b is a realizer of B
$\exists x\, B(x)$	(n, b), where b is a realizer of $B(\bar{n})$.

Kleene's 1945 realizability

a realizer of	**has the form**
$A \to B$	e, where e is the Gödel number of a Turing machine M_e such that M_e halts with a realizer for B whenever a realizer of A is run on M_e.
$\neg A$	any e providing there is **no** realizer for A.
$\forall x\, B(x)$	e, where e is a Gödel number of a Turing machine M_e such that M_e outputs a realizer for $A(\bar{n})$ when run on n.

Basic Assumptions

Let T be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the n-th element of ω.

The Disjunction Property

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1. T has the **disjunction property**, **DP**, if whenever

$$T \vdash \psi \vee \theta$$

holds for sentences ψ and θ of T, then

$$T \vdash \psi \quad \text{or} \quad T \vdash \theta.$$

The Existence Property

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1. T has the **numerical existence property**, **NEP**, if whenever

$$T \vdash (\exists x{\in}\omega)\phi(x)$$

holds for a formula $\phi(x)$ with at most the free variable x, then

$$T \vdash \phi(\bar{n})$$

for some n.

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2. T has the **existence property**, **EP**, if whenever

$$T \vdash \exists x \phi(x)$$

holds for a formula $\phi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that

$$T \vdash \exists! x\, \vartheta(x) \quad \text{and} \quad T \vdash \exists x\, [\vartheta(x) \wedge \phi(x)].$$

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- **Gödel** (1932) observed that intuitionistic propositional logic has the **DP**.
- **Gentzen** (1934): Intuitionistic predicate logic has the **DP** and **EP**.
- **Kleene** (1945): **HA** has the **DP** and **NEP**.
- **Joan Moschovakis** (1965): **DP**, **NEP** and **EP** for (many) systems of intuitionistic analysis.

Remarks about Classical Set Theories

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- Ignoring the trivial counterexamples, classical theories never have the **DP** or the **NEP**.
- **Z** (Zermelo set theory), **ZF**, and **ZF** are known **not** to have the **EP**.
- **ZFC** proves that $\mathbb{R}$ is well-orderable, but it cannot prove that there is a **definable** well-ordering of $\mathbb{R}$.

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  2. (Feferman, Lévy) **EP** fails for Π^1_2 in **ZF** and **ZFC**.
  3. (Y. Moschovakis) **ZF** + Projective Determinacy has the **projective existence property** ($\varphi(x)$, $\vartheta(x)$ projective).

Intuitionistic Zermelo-Fraenkel set theory, **IZF**

* **Extensionality**

Intuitionistic Zermelo-Fraenkel set theory, IZF

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- **Pairing, Union, Infinity**

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- **Full Separation**
- **Powerset**

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- $\#$ **Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x, y) \quad \rightarrow \quad \exists b\ (\forall x \in a)\, (\exists y \in b)\ \varphi(x, y)$$

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Myhill's $\mathbf{IZF}_R$:

IZF with **Replacement** instead of **Collection**

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$\#$ **Strong Collection**

$$(\forall x \in a)\, \exists y\; \varphi(x,y) \;\rightarrow$$
$$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\; \varphi(x,y) \;\wedge\; (\forall y \in b)\, (\exists x \in a)\; \varphi(x,y)\,]$$

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* $*$ **Set Induction scheme**

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- $*$ **Set Induction scheme**

CZF^{-} is **CZF** without Exponentiation.

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- **Explicit set existence axioms**: e.g. Separation, Replacement, Exponentiation
- **Non-explicit** set existence axioms: e.g. in classical set theory Axioms of Choice
- **Non-explicit set existence axioms** in intuitionistic set theory: e.g. Axioms of Choice, (Strong) Collection, Subset Collection, Regular Extension Axiom

Some History

Let $\mathbf{IZF}_R$ result from **IZF** by replacing Collection with Replacement, and let **CST** be Myhill's constructive set theory.

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Theorem 3. (Friedman, Scedrov)
IZF does not have the **EP**.

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The **DP** *and the* **NEP** *hold true for* **CZF**, **CZF** + **REA** *and* **CZF** + *Large Set Axioms.*
One can also add Subset Collection and the following choice principles:

$\mathbf{AC}_\omega$, **DC**, **RDC**, **PAx**.

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- (Beeson 1985) Does any reasonable set theory **with Collection** have the existential definability property?

The Weak Existence Property

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T has the **weak existence property**, **wEP**, if whenever

$$T \vdash \exists x \phi(x)$$

holds for a formula $\phi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that

$$\begin{array}{lll} T & \vdash & \exists! x\, \vartheta(x), \\ T & \vdash & \forall x\, [\vartheta(x) \rightarrow \exists u\, u \in x], \\ T & \vdash & \forall x\, [\vartheta(x) \rightarrow \forall u \in x\, \phi(u)]. \end{array}$$

IZF *and wEP*

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Theorem IZF does not have the weak existence property property.

The Uniform Existence Property

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1. T has the **uniform existence property**, **uEP**, if whenever

$$T \vdash \forall u\, \exists x \phi(u, x)$$

holds for a formula $\phi(u, x)$ having at most the free variable u, x, then there is a formula $\vartheta(u, x)$ with exactly u, x free, so that

$$T \vdash \forall u\, \exists! x\, \vartheta(u, x) \quad \text{and} \quad T \vdash \forall u\, \exists x\, [\vartheta(u, x) \wedge \phi(u, x)].$$

The Uniform Weak Existence Property

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T has the **uniform weak existence property**, **uwEP**, if the following holds: if

$$T \vdash \forall u \exists x A(u, x)$$

holds for a formula $A(u, x)$ having at most the free variables u, x, then there is a formula $B(u, x)$ with exactly u, x free, so that

$$\begin{aligned} T &\vdash \forall u \exists! x\, B(u, x), \\ T &\vdash \forall u \forall x\, [B(u, x) \rightarrow \exists z\, z \in x], \\ T &\vdash \forall u \forall x\, [B(u, x) \rightarrow \forall z \in x\, A(u, z)]. \end{aligned}$$

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- However, we shall introduce an extended notion of E-computability, christened **$E_\wp$-computability**, rendering the function

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- However, we shall introduce an extended notion of E-computability, christened **$E_{\wp}$-computability**, rendering the function

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computable as well.
- Classically, $E_{\wp}$-computability is related to **power recursion**, where the power set operation is regarded to be an initial function. Notion studied by Yiannis Moschovakis and Larry Moss.

Realizability with sets of witnesses

We use the expression $a \neq \emptyset$ to convey the positive fact that the set a is inhabited, that is $\exists x\, x \in a$.
We define a relation

$$a \Vdash_{\mathfrak{wt}} B$$

between sets and set-theoretic formulae.

$$a \bullet f \Vdash_{\mathfrak{wt}} B$$

will be an abbreviation for

$$\exists x[\{a\}(f) \simeq x \ \wedge \ x \Vdash_{\mathfrak{wt}} B]$$

$$
\begin{array}{rcl}
a \Vdash_{\mathfrak{wt}} A & \text{iff} & A \text{ holds true, whenever } A \text{ is an atomic formula} \\
a \Vdash_{\mathfrak{wt}} A \wedge B & \text{iff} & \jmath_0 a \Vdash_{\mathfrak{wt}} A \ \wedge \ \jmath_1 a \Vdash_{\mathfrak{wt}} B \\
a \Vdash_{\mathfrak{wt}} A \vee B & \text{iff} & a \neq \emptyset \ \wedge \ (\forall d \in a)\big(\big[\jmath_0 d = 0 \ \wedge \ \jmath_1 d \Vdash_{\mathfrak{wt}} A\big] \ \vee \\
 & & \qquad \big[\jmath_0 d = 1 \ \wedge \ \jmath_1 d \Vdash_{\mathfrak{wt}} B\big]\big) \\
a \Vdash_{\mathfrak{wt}} \neg A & \text{iff} & \neg A \ \wedge \ \forall c \, \neg c \Vdash_{\mathfrak{wt}} A \\
a \Vdash_{\mathfrak{wt}} A \to B & \text{iff} & (A \to B) \ \wedge \ \forall c \left[c \Vdash_{\mathfrak{wt}} A \ \to \ a \bullet c \Vdash_{\mathfrak{wt}} B\right] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\; a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\, A & \text{iff} & a \neq \emptyset \ \wedge \ (\forall d \in a)[\jmath_0 d \in b \ \wedge \ \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d] \\
a \Vdash_{\mathfrak{wt}} \forall x A & \text{iff} & \forall c \ \ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} \exists x A & \text{iff} & a \neq \emptyset \ \wedge \ (\forall d \in a)\ \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]
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a \Vdash_{\mathfrak{wt}} \neg A & \text{iff} & \neg A \wedge \forall c\, \neg c \Vdash_{\mathfrak{wt}} A \\
a \Vdash_{\mathfrak{wt}} A \rightarrow B & \text{iff} & (A \rightarrow B) \wedge \forall c \big[c \Vdash_{\mathfrak{wt}} A \;\rightarrow\; a \bullet c \Vdash_{\mathfrak{wt}} B\big] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\; a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\, A & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)(\jmath_0 d \in b \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]) \\
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$$\Vdash_{\mathfrak{wt}} B \text{ iff } \exists a\, a \Vdash_{\mathfrak{wt}} B.$$

If we use indices of $E_{\wp}$-recursive functions rather than $E_{\exp}$-recursive functions, we notate the corresponding notion of realizability by $a \Vdash^{\wp}_{\mathfrak{wt}} B$.

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If we use indices of $E_{\wp}$-recursive functions rather than $E_{\exp}$-recursive functions, we notate the corresponding notion of realizability by $a \Vdash^{\wp}_{\mathfrak{wt}} B$.

Corollary

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Corollary

(i) $\mathbf{CZF} \vdash (\Vdash_{\mathfrak{wt}} B) \rightarrow B$.

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Corollary

(i) $\mathbf{CZF} \vdash (\Vdash_{\mathfrak{wt}} B) \rightarrow B$.

(ii) $\mathbf{CZF} + \mathbf{Pow} \vdash (\Vdash^{\wp}_{\mathfrak{wt}} B) \rightarrow B$.

Theorem The theories $\mathbf{CZF}^{-}$, **CZF** and $\mathbf{CZF} + \mathbf{Pow}$ have the uniform weak existence property.

Even better

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- **THEOREM** If

$$\mathbf{CZF} \vdash \exists x\, A(x)$$

then one can effectively construct a $\Sigma^{\mathcal{E}}$ formula $B(y)$ such that

$$\mathbf{CZF} \vdash \exists! y\, B(y)$$

$$\mathbf{CZF} \vdash \forall y [\, B(y) \rightarrow \exists x\ x \in y]$$

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- **THEOREM** If

$$\mathbf{CZF} + \mathbf{Pow} \vdash \exists x\, A(x)$$

then one can effectively construct a $\Sigma^{\mathcal{P}}$ formula $B(y)$ such that

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Power Kripke-Platek Set Theory

We call a formula of $\mathcal{L}_\in$ $\Delta_0^{\mathcal{P}}$ if all its quantifiers are of the form $Q\,x \subseteq y$ or $Q\,x{\in}y$ where Q is $\forall$ or $\exists$ and x and y are distinct variables.

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The $\Delta_0^{\mathcal{P}}$ formulas are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a.$$

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KP($\mathcal{P}$) has the following axioms: Extensionality, Pairing, Union, Infinity, Powerset, $\Delta_0^{\mathcal{P}}$-Separation and $\Delta_0^{\mathcal{P}}$-Collection.

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2. Alternatively, **KP**($\mathcal{P}$) can be obtained from **KP** by adding a function symbol $\mathcal{P}$ for the powerset function as a primitive symbol to the language and the axiom

$$\forall y\,[y \in \mathcal{P}(x) \leftrightarrow y \subseteq x]$$

   and extending the schemes of Δ_0 Separation and Collection to the Δ_0 formulae of this new language.

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1. **KP**$(\mathcal{P})$ is **not** the same as **KP** $+$ Powerset. The latter is a much weaker theory in which one cannot prove the existence of $V_{\omega+\omega}$.
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3. The **power admissible** sets are the transitive models of **KP**$(\mathcal{P})$.

Intuitionistic Power Kripke-Platek Set Theory and friends

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Intuitionistic Power Kripke-Platek Set Theory and friends

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IKP($\mathcal{E}$) is the intuitionistic theory with the axioms: Extensionality, Pairing, Union, Infinity, Exponentiation, $\Delta_0^{\mathcal{E}}$-Separation and $\Delta_0^{\mathcal{E}}$-Collection.

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Proof: All these theories have explicit set existence axioms. Using techniques developed by J. Moschovakis, H. Friedman and J. Myhill, this can be proved. □

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Theorem 4: (**A. Swan**) **CZF** + Subset Collection does **not** have the **weak existence property**.

Ordinal Analysis of Set Theories

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  KPi $=$ **KP** $+$ Every set is contained in an admissible set.

  Corresponds to $\Delta^1_2\text{-CA} + $ bar induction

Henceforth Ω will be a name for a large ordinal or even the whole class of ordinals.

The problem of “naming” sets will be solved by building a formal von Neumann hierarchy using the ordinals $< \Omega$.

Power terms

Definition. We define the $RS_{\Omega}^{\mathcal{P}}$–terms. To each $RS_{\Omega}^{\mathcal{P}}$–term t we also assign its *rank*, $|t|$.

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2. For each $\alpha < \Omega$, we have infinitely many free variables $a_1^\alpha, a_2^\alpha, a_3^\alpha, \ldots$ which are $RS_\Omega^{\mathcal{P}}$–term with $|\,a_i^\alpha\,| = \alpha$.

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3. If $F(x, \vec{y}\,)$ is a $\Delta_0^{\mathcal{P}}$ formula (whose free variables are among those indicated) and $\vec{s} \equiv s_1, \cdots, s_n$ are $RS_{\Omega}^{\mathcal{P}}$–terms, then the formal expression

$$\{x{\in}\mathbb{V}_{\alpha} \mid F(x, \vec{s})\}$$

is an $RS_{\Omega}^{\mathcal{P}}$–term with $|\,\{x{\in}\mathbb{V}_{\alpha} \mid F(x, \vec{s})\}\,| = \alpha$.

$RS_\Omega^{\mathcal{P}}$*–formulae*

The $RS_\Omega^{\mathcal{P}}$*–formulae* are the expressions of the form

$$F(s_1, \ldots, s_n),$$

where $F[a_1, \ldots, a_n]$ is a formula of $\mathbf{KP}(\mathcal{P})$ and $s_1, \ldots, s_n$ are $RS_\Omega^{\mathcal{P}}$-terms.

For technical convenience, we let $\neg A$ be the formula which arises from A by (i) putting $\neg$ in front of each atomic formula, (ii) replacing $\wedge, \vee, (\forall x \in a), (\exists x \in a), \forall x, \exists x$ by $\vee, \wedge, (\exists x \in a), (\forall x \in a), \exists x, \forall x$, respectively, and (iii) dropping double negations.

References

1. R.: **The disjunction and related properties for constructive Zermelo-Fraenkel set theory.** Journal of Symbolic Logic 70 (2005) 1233–1254.
2. R.: **Metamathematical Properties of Intuitionistic Set Theories with Choice Principles.** In: S. B. Cooper, B. Löwe, A. Sorbi (eds.): *New Computational Paradigms: Changing Conceptions of What is Computable* (Springer, New York, 2008) 287–312.
3. R.: **From the weak to the strong existence property.** Annals of Pure and Applied Logic doi:10.1016/j.apal.2012.01.012
4. R.: **Relativized ordinal analysis: The case of Power Kripke-Platek set theory,** Proceedings of the Infinity Project, CRM Documents vol. 11, Centre de Recerca Matemàtica, Bellaterra (Barcelona) (2012).